[This question paper contains 6 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1395 A

Unique Paper Code : 32351403

Name of the Paper : BMATH-410 – Ring Theory and Linear Algebra – I

Name of the Course : CBCS (LOCF) B.Sc. (H) Mathematics

Semester : IV

Duration : 3.30 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any two parts from each question.
3. All questions are compulsory.

1. (a) Define zero divisors in a ring. Let R be the set of all real valued functions defined for all real numbers under function addition and multiplication. Determine all zero divisors of R. (6½)

P.T.O.

(b) What is nilpotent element? If a and b are nilpotent elements of a commutative ring, show that a + b is also nilpotent. Give an example to show that this may fail if the ring R is not commutative.

(6½)

(c) Let R be a commutative ring with unity. Prove that U(R), the set of all units of R, form a group under multiplication of R. (6½)

(d) Determine all subrings of $\mathbb{Z}$, the set of integers.

(6½)

2. (a) Define centre of a ring. Prove that centre of a ring R is a subring of R. (6)

(b) Suppose R is a ring with $a^2 = a$, for all $a \in R$. Show that R is a commutative ring. (6)

(c) Show that any finite field has order p^n, where p is prime. (6)

(d) Let R be a ring with unity 1. Prove that if 1 has infinite order under addition, then CharR = 0, and if 1 has order n under addition, then CharR = n.

(6)

3. (a) Let R be a commutative ring with unity and let A be an ideal of R. Then show that R/A is a field if and only if A is maximal ideal. (6½)

(b) Prove that I = < 2 + 2i > is not prime ideal of $\mathbb{Z}[i]$.

How many elements are in $\frac{\mathbb{Z}[i]}{I}$? What is the characteristic of $\frac{\mathbb{Z}[i]}{I}$? (6½)

(c) In $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, let $I = \{f(x) \in \mathbb{Z}[x] \mid f(0) = 0\}$. Prove that I is not a maximal ideal. (6½)

(d) Let $\mathbb{R}[x]$ denote the ring of polynomials with real coefficients and let $< x^2 + 1 >$ denote the principal ideal generated by $x^2 + 1$. Then show that

$$\frac{\mathbb{R}[x]}{\langle x^2+1\rangle} = \{g(x) + \langle x^2+1\rangle \mid g(x) \in \mathbb{R}[x]\} =$$

$$\{ax + b + \langle x^2+1\rangle \mid a, b \in \mathbb{R}\}$$

(6½)

P.T.O.

4. (a) If R is a ring with unity and the characteristic of R is $n > 0$, then show that R contains a subring isomorphic to $\mathbb{Z}_n$ and if the characteristic of R is 0 then R contains a subring isomorphic to $\mathbb{Z}$. (6)

(b) Determine all ring homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$. (6)

(c) Let n be an integer with decimal representation $a_k a_{k-1} \dots a_1 a_0$. Prove that n is divisible by 11 if and only if $a_0 - a_1 + a_2 - \dots + (-1)^k a_k$ is divisible by 11. (6)

(d) Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism. (6)

5. (a) Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 + W_2$ is a smallest subspace of V that contains both W_1 and W_2. (6)

(b) For the following polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as linear combination of other two. (6)

$$\{x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3\}.$$

(c) Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some $k \, (1 \leq k < n)$. (6)

(d) Let $W_1 = \{(a, b, 0) \in \mathbb{R}^3 : a, b \in \mathbb{R}\}$ and $W_2 = \{(0, b, c) \in \mathbb{R}^3 : b, c \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Determine $\dim(W_1)$, $\dim(W_2)$, $\dim(W_1 \cap W_2)$ and $\dim(W_1 + W_2)$. Hence deduce that $W_1 + W_2 = \mathbb{R}^3$. Is $\mathbb{R}^3 = W_1 \oplus W_2$? (6)

6. (a) Let V and W be finite-dimensional vector spaces having ordered bases β and γ respectively, and let $T : V \to W$ be linear. Then for each $u \in V$, show

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta.$$ (6½)

(b) Let $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be linear transformation defined by

$$T(a + bx + cx^2) = (a - c) + (a - c)x + (b - a)x^2 + (c - b)x^3.$$

Find null space N(T) and range space R(T). Also verify Rank-Nullity Theorem. (6½)

P.T.O.